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Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities

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Abstract

Multiple critical points theorems for non-differentiable functionals are established. Applications both to elliptic variational–hemivariational inequalities and eigenvalue problems with discontinuous nonlinearities are then presented.

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1. Introduction

The critical point theory for non-smooth functionals, expressed as a sum of a locally Lipschitz function and a convex, proper, and lower semicontinuous function, has been developed by D. Motreanu and P.D. Panagiotopoulos (see [31, Chapter 3] and the references therein). It contains the theory for locally Lipschitz functionals investigated by K.C. Chang [16], which is

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[1] This paper is devoted to the memory of my wife Roberta Salvati, who died from cancer on the 27th of July 2006. Her smile is in my soul, for ever and ever.

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based on the Nonsmooth Analysis by F.H. Clarke [17], and generalizes the study on the variational inequalities as given by A. Szulkin [39]. This sort of theory for functionals of the above mentioned type arises in several mechanical and engineering questions which lead us to consider functionals lacking smoothness properties and to study variational–hemivariational inequalities (see, for instance, [33] and [31, Chapter 3]).

In this framework, very recently, S.A. Marano and D. Motreanu ([29] and [30]) have established multiple critical points theorems, which extend the results previously obtained by B. Ricceri ([35] and [36]) for differentiable functionals to non-smooth functionals.

The main aim of the present paper is to establish multiple critical points theorems for non-smooth functionals (Theorems 3.1–3.3) that improve the results in [29] and [30] and, consequently, in [35] and [36] (see Remarks 3.3, 3.6 and 3.8). In particular, we point out here that, contrary to [29, Theorem B], in Theorem 3.3 the coercivity assumption on the functional is not required and a more precise estimate of the real parameter is determined (Remark 3.8). Moreover, it is worth noting that the proof of Theorem 3.1 is completely different with respect to the proof of [30, Theorem 1.1(a)] and, in addition, ensures a more precise result (Remark 3.3).

As an application of results in Section 3, a variational–hemivariational inequality depending on a real parameter is investigated (Theorem 4.1) and some remarks on the growth of the function and on the values of the real parameter are made (Remark 4.1). The main result in Section 4 is Theorem 4.2 that ensures three weak solutions to elliptic Dirichlet problems

$$\begin{cases} \Delta u = \lambda\big(f(x,u) + \mu g(x,u)\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$D_{\lambda,\mu}$}$$

where Ω is a non-empty bounded open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, λ and μ are suitable real positive constants, and f and g are functions that may be discontinuous with respect to u. Problems of this type have been studied by looking for solutions of the corresponding differential inclusion obtained by filling the gaps at the discontinuity points of f and g with respect to u (see, for instance, [20,22] and the references therein). On the contrary, Theorem 4.2 and its consequence, that is, Theorem 4.3, ensure solutions that are actually weak solutions for problem $(D_{\lambda,\mu})$; nevertheless, the set of discontinuity points may also be uncountable (see Remark 4.4 and Example 4.2). Moreover, Theorem 4.2 improves [29, Theorem 4.2] since any growth condition on g is allowed (see Remark 4.3 and Example 4.1). Further, it is worth noting that Theorems 4.2 and 4.3 are novel results also when f and g are continuous functions. In particular, Theorem 4.2 improves [36, Theorem 4] when f and g are non-positive functions, and is mutually independent from the results in [26,27] and [34] (see Remark 4.5 and Example 4.3).

Finally, as a further example of applications of the results in Section 3, we obtain three weak solutions to the autonomous Dirichlet problem involving the p-Laplacian, with $p > N$ (Theorem 4.4). By way of example, here, denoting with D, K and R three known positive constants depending on p, N and Ω (see (4.3), (4.16) and (4.17) in Section 4), we present a simple consequence of Theorem 4.4.

Theorem 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a positive, bounded, and almost everywhere continuous function. Assume that there exist two positive constants c and d, with $c < Kd$, such that*

$$\frac{1}{c^p}\int_0^c f(t)\,dt < \frac{R}{2}\frac{1}{d^p}\int_0^d f(t)\,dt.$$

Then, for each $\lambda \in \left]\frac{2^{p+1}(2^N-1)}{pD^p}\frac{d^p}{\int_0^d f(t)\,dt}; \frac{1}{m(\Omega)pK^p}\frac{c^p}{\int_0^c f(t)\,dt}\right[$, *the problem*

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

possesses at least three weak solutions.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. We denote the dual space of X by X^*, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $\Theta : X \to \mathbb{R}$ is called locally Lipschitz when to every $u \in X$ there corresponds a neighbourhood U of u and a constant $L \geqslant 0$ such that

$$\left|\Theta(v) - \Theta(w)\right| \leqslant L\|v - w\| \quad \text{for all } v, w \in U.$$

If $u, v \in X$, the symbol $\Theta^\circ(u; v)$ indicates the generalized directional derivative of Θ at point u along direction v, namely

$$\Theta^\circ(u; v) := \limsup_{w \to u,\, t \to 0^+} \frac{\Theta(w + tv) - \Theta(w)}{t}.$$

The generalized gradient of the functional Θ at u, denoted by $\partial\Theta(u)$, is the set

$$\partial\Theta(u) := \left\{u^* \in X^*\colon\ \langle u^*, v\rangle \leqslant \Theta^\circ(u; v) \text{ for all } v \in X\right\}.$$

Given a proper, convex and lower semicontinuous function $j : X \to]-\infty, +\infty]$, the set $D(j) = \{u \in X\colon\ j(u) < +\infty\}$ denotes its effective domain.

Let $I : X \to]-\infty, +\infty]$ fulfills the following structure hypothesis

(H) $I = \Theta + j$ *where* $\Theta : X \to \mathbb{R}$ *is locally Lipschitz while* $j : X \to]-\infty, +\infty]$ *is convex, proper and lower semicontinuous.*

Critical points of I are defined as the solutions to the following problem

$$\Theta^\circ(u; v - u) + j(v) - j(u) \geqslant 0 \quad \text{for all } v \in X.$$

Moreover, in the present non-smooth setting, we say that functional I verifies the *Palais–Smale condition at level* c, $c \in \mathbb{R}$ (in short $(PS)_c$) if any sequence $\{u_n\}$ such that

(α) $I(u_n) \to c$, $c \in \mathbb{R}$,
(β) $\Theta^\circ(u_n; v - u_n) + j(v) - j(u_n) \geqslant -\epsilon_n\|v - u_n\|$ for all $v \in X$, where $\epsilon_n \to 0^+$,

has a convergent subsequence.

It is worth noting that for $\Theta \in C^1(X, \mathbb{R})$ the above definitions reduce to those of Szulkin [39]. When $j \equiv 0$ they coincide with the corresponding definitions of Chang [16]. For a thorough treatment of these topics we refer to [17,31,32] and the references therein.

Finally, let $\Phi, \Upsilon : X \to \mathbb{R}$ be two locally Lipschitz functionals and let $M > 0$. Put

$$\Upsilon_M(u) = \begin{cases} \Upsilon(u) & \text{if } \Upsilon(u) \leqslant M, \\ M & \text{if } \Upsilon(u) > M. \end{cases}$$

Clearly, $\Theta_M = \Phi - \Upsilon_M$ is a locally Lipschitz functional. We say that $\Phi - \Upsilon$ verifies the *Palais–Smale condition at level c, $c \in \mathbb{R}$, cut off at M* (in short $(PS)_c^M$) if Θ_M satisfies $(PS)_c$ condition.

3. Multiple critical points theorems

In this section X is a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive real parameter, $j : X \to]-\infty, +\infty]$ is a convex, proper and lower semicontinuous functional and $D(j)$ is the effective dominion of j. Write

$$\Psi := \Upsilon - j \quad \text{and} \quad I_\lambda := \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j.$$

We also assume that Φ is coercive and

$$D(j) \cap \Phi^{-1}\big(]-\infty, r[\big) \neq \emptyset \tag{1}$$

for all $r > \inf_X \Phi$. Moreover, owing to (1) and provided $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$, we can define

$$\varphi^{(1)}(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\varphi^{(2)}(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \sup_{v \in \Phi^{-1}([r, +\infty[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\varphi_1(r_1, r_2) = \max\big\{\varphi^{(1)}(r_1); \varphi^{(1)}(r_2)\big\},$$

$$\varphi_2(r_1, r_2) = \inf_{u \in \Phi^{-1}(]-\infty, r_1[)} \sup_{v \in \Phi^{-1}([r_1, r_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}.$$

We specify that in the definitions of $\varphi^{(2)}$ and φ_2 we read $\Psi(v) - \Psi(u)$ as $\Upsilon(v) - \Upsilon(u)$ when both v and u are not in $D(j)$. Moreover, $\varphi_1(r_1, r_2)$ could be 0; in this and similar cases, in the sequel, we agree to read $\frac{1}{\varphi_1(r_1, r_2)}$ as $+\infty$.

We have the following result.

Theorem 3.1. *Assume that there are $r_1, r_2 \in \mathbb{R}$, with $\inf_X \Phi < r_1 < r_2$, such that*

(a) $\varphi_1(r_1, r_2) < \varphi_2(r_1, r_2)$.

Then, for each $\lambda \in \Lambda_{r_1, r_2} :=]\frac{1}{\varphi_2(r_1, r_2)}, \frac{1}{\varphi_1(r_1, r_2)}[$ the restriction of the functional I_λ to $\Phi^{-1}(]-\infty, r_1[)$ admits a global minimum u_1 and the restriction of the functional I_λ to $\Phi^{-1}(]-\infty, r_2[)$ admits a global minimum $u_2 \notin \Phi^{-1}(]-\infty, r_1[)$.

Proof. Fix $\lambda \in \Lambda_{r_1,r_2}$. We claim that there is $u_1 \in D(j) \cap \Phi^{-1}(]-\infty, r_1[)$ such that $I_\lambda(u_1) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r_1[)$. Since $\frac{1}{\lambda} > \varphi^{(1)}(r_1)$, there is $\bar{u} \in D(j)$ such that $\Phi(\bar{u}) < r_1$ and $\Phi(\bar{u}) - \lambda\Psi(\bar{u}) < r_1 - \lambda \sup_{\Phi(x)<r_1} \Psi(x)$. Moreover, put $M = \frac{r_1 - \Phi(\bar{u})}{\lambda} + \Psi(\bar{u})$. Clearly,

$$\sup_{\Phi(x)<r_1} \Psi(x) < M. \tag{3.1}$$

Finally, put

$$\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) \leqslant M, \\ M & \text{if } \Psi(u) > M. \end{cases}$$

Since, owing to [15, Corollary III.8] j is sequentially weakly lower semicontinuous, a simple computation shows that Ψ_M is sequentially weakly upper semicontinuous. Put $J = \Phi - \lambda\Psi_M$. J is a sequentially weakly lower semicontinuous functional and, as it is easy to see, it is also a coercive functional. Therefore (see, for instance, [38, Theorem 1.2]), it admits a global minimum u_0. If $J(u_0) = J(\bar{u})$, we take $u_1 = \bar{u}$ and obtain the conclusion. Otherwise, assume $J(u_0) < J(\bar{u})$. In this last case, we have that $\Psi(u_0) < M$. In fact, from $J(u_0) < J(\bar{u})$ one has $\Phi(u_0) - \lambda\Psi_M(u_0) < \Phi(\bar{u}) - \lambda\Psi_M(\bar{u})$. Hence, $\Phi(u_0) < \lambda\Psi_M(u_0) + \Phi(\bar{u}) - \lambda\Psi(\bar{u}) \leqslant \lambda M + \Phi(\bar{u}) - \lambda\Psi(\bar{u}) = r_1$ and, from (3.1) one has $\Psi(u_0) < M$. Therefore, $\Phi(u_0) - \lambda\Psi(u_0) = \Phi(u_0) - \lambda\Psi_M(u_0) \leqslant \Phi(u) - \lambda\Psi_M(u)$ for all $u \in X$ and, taking again (3.1) into account, $\Phi(u_0) - \lambda\Psi(u_0) \leqslant \Phi(u) - \lambda\Psi(u)$ for all $u \in \Phi^{-1}(]-\infty, r_1[)$. Hence, taking $u_1 = u_0$, our claim is proved.

Now, arguing as before and taking into account that $\frac{1}{\lambda} > \varphi^{(1)}(r_2)$, we obtain that there is $u_2 \in D(j) \cap \Phi^{-1}(]-\infty, r_2[)$ such that $I_\lambda(u_2) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r_2[)$.

Finally, we prove that $u_2 \notin \Phi^{-1}(]-\infty, r_1[)$. Indeed, if $u_2 \in \Phi^{-1}(]-\infty, r_1[)$, from $\frac{1}{\lambda} < \varphi_2(r_1, r_2)$ there is $\bar{v} \in \Phi^{-1}([r_1, r_2[)$ such that $\frac{1}{\lambda} < \frac{\Psi(\bar{v}) - \Psi(u_2)}{\Phi(\bar{v}) - \Phi(u_2)}$, that is $\Phi(\bar{v}) - \lambda\Psi(\bar{v}) < \Phi(u_2) - \lambda\Psi(u_2)$ and this is absurd since u_2 is a global minimum in $\Phi^{-1}(]-\infty, r_2[)$. Hence, the proof is complete. □

Remark 3.1. Clearly, inequality (a) in Theorem 3.1 signifies the following two inequalities

(a_1) $\varphi^{(1)}(r_1) < \varphi_2(r_1, r_2)$;
(a_2) $\varphi^{(1)}(r_2) < \varphi_2(r_1, r_2)$.

In Remark 3.11 easy inequalities that imply (a_1) and (a_2) are pointed out.

Now, here and in the sequel we also assume that Φ and Υ are locally Lipschitz functionals.

Remark 3.2. Theorem 3.1, taking into account that now Φ is a continuous functional and that each local minimum is actually a critical point of I_λ (see Proposition 2.1 of [29]), ensures the existence of two distinct critical points.

Remark 3.3. Fix $r > \inf_X \Phi$. From the proof of Theorem 3.1, in particular, we obtain that, for each $\lambda \in \left]0, \frac{1}{\varphi^{(1)}(r)}\right[$, the functional I_λ admits a critical point which lies in $\Phi^{-1}(]-\infty, r[)$.

In [30, Theorem 1.1(a)] the same result for each $\lambda \in]0, \frac{1}{\underline{\varphi}^{(1)}(r)}[$ was proved, where

$$\underline{\varphi}^{(1)}(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{u \in \overline{(\Phi^{-1}]-\infty, r[)}^{w}} \Psi(u)) - \Psi(u)}{r - \Phi(u)}$$

and $\overline{(\Phi^{-1}] - \infty, r[)}^{w}$ is the closure of $\Phi^{-1}(]-\infty, r[)$ in the weak topology on X.

Since $\varphi^{(1)}(r) \leqslant \underline{\varphi}^{(1)}(r)$, our result, with a completely different proof, improves [30, Theorem 1.1(a)].

For the same reasons, our result improves [35, Theorem 2.5(a)]. We also observe that, with the same remarks as above, Theorem 3.1 extends [8, Theorem 2.1] to non-smooth functionals.

We have the following three critical points theorem

Theorem 3.2. *Assume that there is $r_1 \in \mathbb{R}$, with $\inf_X \Phi < r_1$, such that*

(a_1') $\varphi^{(1)}(r_1) < \varphi^{(2)}(r_1)$.

Assume also that for each $\lambda \in \Lambda^{r_1} :=]\frac{1}{\varphi^{(2)}(r_1)}, \frac{1}{\varphi^{(1)}(r_1)}[$ one has

(b_1) *the functional $\Phi - \lambda\Psi$ is bounded below and fulfills $(PS)_c$, $c \in \mathbb{R}$.*

Then, for each $\lambda \in \Lambda^{r_1}$, the functional I_λ admits at least three distinct critical points.

Proof. Fix $\lambda \in \Lambda^{r_1}$. We note at once that, taking [15, Corollary III.8] into account, the functional I_λ is sequentially weakly lower semicontinuous and, from (b_1), taking [32, Corollary 1.3] into account, it is also coercive. Therefore, I_λ admits a global minimum that we call u_1. On the other hand, taking into account that $\frac{1}{\lambda} > \varphi^{(1)}(r_1)$ and arguing as in the proof of Theorem 3.1, the restriction of I_λ to $\Phi^{-1}(]-\infty, r_1[)$ admits a global minimum that we call u_2.

Now, we prove that $u_1 \notin \Phi^{-1}(]-\infty, r_1[)$. Indeed, if $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, from $\frac{1}{\lambda} < \varphi^{(2)}(r_1)$ there is $\bar{v} \in \Phi^{-1}([r_1, +\infty[)$ such that $\frac{1}{\lambda} < \frac{\Psi(\bar{v}) - \Psi(u_1)}{\Phi(\bar{v}) - \Phi(u_1)}$, that is $\Phi(\bar{v}) - \lambda\Psi(\bar{v}) < \Phi(u_1) - \lambda\Psi(u_1)$ and this is absurd since u_1 is a global minimum of I_λ.

Finally, Corollary 2.1 of [29] ensures the conclusion. □

Remark 3.4. It is a simple matter to show that inequality (a_1') is equivalent to inequality (a_1).

Remark 3.5. If we assume that (a) of Theorem 3.1 holds and that, for each $\lambda \in \Lambda_{r_1,r_2}$, one has

(b_2) the functional $\Phi - \lambda\Psi$ fulfills $(PS)_c$, $c \in \mathbb{R}$,

then, owing to Corollary 2.1 of [29], the functional I_λ, for each $\lambda \in \Lambda_{r_1,r_2}$, admits at least three distinct critical points.

With respect to Theorem 3.2, in this case the boundedness below of I_λ is not requested. On the contrary, in Theorem 3.2, the condition *there is $r_2 > r_1$ such that (a_2) holds* is not requested.

Remark 3.6. With respect to the three critical points in [29, Theorem 3.1] (see also [10, Theorem 2.1]), we obtain an interval of parameters, λ, for which the functional has three critical points, which is precisely determined.

We also observe that Theorem 3.2 extends [2, Theorem 2.1] (see also Remark 3.3) to non-smooth functionals.

When $j \equiv 0$, we can give a variant of Theorem 3.2 assuming $(PS)_c^M$, $c \in \mathbb{R}$, $M > 0$, instead of $(PS)_c$, $c \in \mathbb{R}$. For this end, we assume that

(2) Φ is convex;
(3) for every x_1, x_2 such that $\Upsilon(x_1) \geqslant 0$ and $\Upsilon(x_2) \geqslant 0$ one has

$$\inf_{t\in[0,1]} \Upsilon\big(tx_1 + (1-t)x_2\big) \geqslant 0;$$

(4) $\inf_X \Phi = \Phi(0) = \Upsilon(0) = 0$.

Moreover, given $r_3 > 0$, we define

$$\varphi^{(3)}(r_2, r_3) = \frac{\sup_{x\in\Phi^{-1}(]-\infty, r_2+r_3[)} \Upsilon(x)}{r_3}$$

and

$$\varphi_3(r_1, r_2, r_3) = \max\big\{\varphi_1(r_1, r_2); \varphi^{(3)}(r_2, r_3)\big\}.$$

Theorem 3.3. *Assume that there are three positive constants r_1, r_2, r_3, with $r_1 < r_2$, such that*

($\underline{a}$) $\varphi_3(r_1, r_2, r_3) < \varphi_2(r_1, r_2)$.

Assume also that for each $\lambda \in \Lambda_{r_1,r_2,r_3} :=]\frac{1}{\varphi_2(r_1,r_2)}, \frac{1}{\varphi_3(r_1,r_2,r_3)}[$ one has

(b_3) *the functional $\Phi - \lambda\Upsilon$ fulfills $(PS)_c^{\frac{r_3}{\lambda}}$, $c \in \mathbb{R}$.*

Then, for each $\lambda \in \Lambda_{r_1,r_2,r_3}$ the functional I_λ admits three critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, $u_2 \in \Phi^{-1}([r_1, r_2[)$ and $u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

Proof. Fix $\lambda \in \Lambda_{r_1,r_2,r_3}$. Owing to Theorem 3.1 the restriction of I_λ to $\Phi^{-1}(]-\infty, r_1[)$ admits a global minimum that we call u_1, which is a local minimum of I_λ, and the restriction of I_λ to $\Phi^{-1}(]-\infty, r_2[)$ admits a global minimum that we call u_2, which is a local minimum of I_λ belonging to $\Phi^{-1}([r_1, r_2[)$. We explicitly observe that

$$I_\lambda(u_2) < I_\lambda(u_1).$$

In fact, from $\frac{1}{\lambda} < \varphi_2(r_1, r_2)$ there is $\overline{v} \in \Phi^{-1}([r_1, r_2[)$ such that $\Phi(\overline{v}) - \lambda\Upsilon(\overline{v}) < \Phi(u_1) - \lambda\Upsilon(u_1)$, so that $I_\lambda(u_2) \leqslant I_\lambda(\overline{v}) < I_\lambda(u_1)$. We may suppose, without loss of generality, $u_1 = 0$ and $I_\lambda(u_1) = 0$.

Now, put

$$\Upsilon_{\frac{r_3}{\lambda}}(u) = \begin{cases} \Upsilon(u) & \text{if } \Upsilon(u) \leqslant \frac{r_3}{\lambda}, \\ \frac{r_3}{\lambda} & \text{if } \Upsilon(u) > \frac{r_3}{\lambda}, \end{cases}$$

and

$$J_\lambda(u) = \Phi(u) - \lambda \Upsilon_{\frac{r_3}{\lambda}}(u).$$

It is a simple computation to show that also $\Upsilon_{\frac{r_3}{\lambda}}$ is a locally Lipschitz function and that, since $\frac{1}{\lambda} > \varphi^{(3)}(r_2, r_3)$, one has

$$\sup_{x \in \Phi^{-1}(]-\infty, r_2+r_3[)} \lambda \Upsilon(x) < r_3. \tag{3.2}$$

Let $\rho > 0$ such that $\overline{B}_\rho \subset \Phi^{-1}(]-\infty, r_1[)$. Clearly $\|u_2\| > \rho$. Moreover, one has $J_\lambda(u_1) \leqslant J_\lambda(u)$ for all $u \in \partial B_\rho$. Owing to (b_3) and taking into account that Φ is coercive, the functional J_λ satisfies $(PS)_c$. Therefore, taking $a = \inf_{\partial B_\rho} J_\lambda \geqslant 0$, Theorem 2.2 of [29] ensures that J_λ has a critical point u_3 such that $J_\lambda(u_3) = c$ and $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\lambda(\gamma(t))$, where $\Gamma = \{\gamma \in C^0([0,1], X)\colon\ \gamma(0) = 0,\ \gamma(1) = u_2\}$. Hence, since (2)–(4) hold and u_2 is a global minimum in $\Phi^{-1}(]-\infty, r_2[)$, one has $J_\lambda(u_3) \leqslant \sup_{t\in[0,1]} J_\lambda(tu_2) = \sup_{t\in[0,1]}[\Phi(tu_2) - \lambda \Upsilon_{\frac{r_3}{\lambda}}(tu_2)] \leqslant \sup_{t\in[0,1]}[t\Phi(u_2)] - \lambda \inf_{t\in[0,1]} \Upsilon_{\frac{r_3}{\lambda}}(tu_2) < r_2$. Hence

$$\Phi(u_3) < r_2 + r_3. \tag{3.3}$$

Therefore, u_3 is a critical point of J_λ which satisfies (3.3). From (3.2) one has that u_3 is also a critical point of I_λ, so the proof is complete. □

Remark 3.7. Clearly, inequality $(\underline{a})$ in Theorem 3.3 signifies three inequalities (a_1), (a_2) (see Remark 3.1) and

(a_3) $\varphi^{(3)}(r_2, r_3) < \varphi_2(r_1, r_2)$.

In Remark 3.11 an easy inequality that implies (a_3) is pointed out.

Remark 3.8. With respect to Theorem 3.2, Theorem 3.3 requires $(PS)_c^M$ instead of $(PS)_c$ and, in addition, ensures that the three critical points are uniformly bounded in norm with respect to parameter λ.

With respect to [29, Theorems 3.1 and B] (see also [10, Theorem 2.1]), the coercivity of I_λ is not requested and a precisely determined interval of parameters λ for which the functional has three critical points, is obtained. We also observe that the key assumption of [29, Theorem 3.1] is a suitable minimax inequality which is equivalent to an inequality of the similar type of (a_1) (see [5–7]).

We also observe that Theorem 3.3 extends [3, Theorem 5.1] (see also Remark 3.3) to non-smooth functionals.

Remark 3.9. Taking the proof of Theorem 3.3 into account, the assumption (3) can be substituted by the following more general condition:

(3′) for each $\lambda \in \Lambda_{r_1,r_2,r_3}$ and for every x_1, x_2, which are local minima for the functional $\Phi - \lambda\Upsilon$, and such that $\Upsilon(x_1) \geqslant 0$ and $\Upsilon(x_2) \geqslant 0$ one has

$$\inf_{t\in[0,1]} \Upsilon\big(tx_1 + (1-t)x_2\big) \geqslant 0.$$

We point out one of the consequences of Theorem 3.3.

Corollary 3.1. *Assume that there are two positive constants ρ_1, ρ_2 and $\overline{v} \in X$, with $\rho_1 < \Phi(\overline{v}) < \rho_2/2$, such that*

(a_1'') $\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}{\rho_1} < \frac{1}{2}\frac{\Upsilon(\overline{v})}{\Phi(\overline{v})}$;

(a_2') $\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)} \Upsilon(u)}{\rho_2} < \frac{1}{4}\frac{\Upsilon(\overline{v})}{\Phi(\overline{v})}$.

Assume also that for each

$$\lambda \in \Lambda'_{\rho_1,\rho_2,\overline{v}} := \left]\frac{2\Phi(\overline{v})}{\Upsilon(\overline{v})}, \min\left\{\frac{\rho_1}{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}; \frac{\rho_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)} \Upsilon(u)}\right\}\right[$$

one has

(b_3) *the functional $\Phi - \lambda\Upsilon$ fulfills $(PS)_c^{\frac{\rho_2}{2\lambda}}$, $c \in \mathbb{R}$.*

Then, for each $\lambda \in \Lambda'_{\rho_1,\rho_2,\overline{v}}$ the functional I_λ admits three critical points u_1, u_2, u_3 which lie in $\Phi^{-1}(]-\infty, \rho_2[)$.

Proof. Put $r_1 = \rho_1$, $r_2 = r_3 = \rho_2/2$. Clearly, one has

$$\varphi^{(1)}(r_1) \leqslant \frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}{\rho_1},$$

$$\varphi^{(1)}(r_2) \leqslant \frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2/2[)} \Upsilon(u)}{\rho_2/2} \leqslant \frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)} \Upsilon(u)}{\rho_2/2},$$

$$\varphi^{(3)}(r_2, r_3) = \frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)} \Upsilon(u)}{\rho_2/2}.$$

Hence

$$\varphi_3(r_1, r_2, r_3) \leqslant \max\left\{\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}{\rho_1}; 2\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)} \Upsilon(u)}{\rho_2}\right\}. \tag{3.4}$$

On the other hand, taking into account that from (a_1'') one has $\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u) < \Upsilon(\overline{v})$, and since $\Phi(u) \geqslant 0$ for all $u \in X$, we obtain

$$\varphi_2(r_1,r_2) \geqslant \frac{\Upsilon(\bar{v}) - \sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}{\Phi(\bar{v})} \geqslant \frac{\Upsilon(\bar{v})}{\Phi(\bar{v})} - \frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}{\Phi(\bar{v})}$$
$$\geqslant \frac{\Upsilon(\bar{v})}{\Phi(\bar{v})} - \frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_1[)} \Upsilon(u)}{\rho_1}.$$

Hence, from (a_1''), one has

$$\varphi_2(r_1,r_2) \geqslant \frac{1}{2}\frac{\Upsilon(\bar{v})}{\Phi(\bar{v})}. \tag{3.5}$$

Therefore, taking (a_1'') and (a_2') into account, Theorem 3.3 ensures the conclusion. □

Remark 3.10. We explicitly observe that when Φ and Υ are regular enough, then (b_3) is satisfied. To be precise, if $\Phi, \Upsilon : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals, $\Phi' : X \to X^*$ admits a continuous inverse operator on X^*, and $\Upsilon' : X \to X^*$ is compact, then, owing to [3, Theorem 3.1], the functional $\Phi - \Upsilon$ satisfies the $(PS)_c^M$ condition for all $M > 0$.

Remark 3.11. Put

$$\chi(r) = \sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)$$

for all $r > \inf_X \Phi$ and assume

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0. \tag{4'}$$

As seen in the proof of Corollary 3.1, the inequality

$$\frac{\chi(r_1)}{r_1} < \frac{1}{2}\frac{\Psi(\bar{v})}{\Phi(\bar{v})},$$

with $\Phi(\bar{v}) > r_1$, implies (a_1) (that is (a_1')); the inequality

$$\frac{\chi(r_2)}{r_2} < \frac{1}{2}\frac{\Psi(\bar{v})}{\Phi(\bar{v})},$$

with $r_1 < \Phi(\bar{v}) \leqslant r_2$, implies (a_2); and, finally, the inequality

$$\frac{\chi(r_2+r_3)}{r_3} < \frac{1}{2}\frac{\Psi(\bar{v})}{\Phi(\bar{v})},$$

with $r_1 < \Phi(\bar{v}) \leqslant r_2$ and $r_3 > 0$, implies (a_3).

Therefore, to apply Theorems 3.1–3.3 easily to nonlinear differential problems, it is enough to estimate an upper bound of the function $\chi(r)$ for some $r > 0$.

Remark 3.12. We explicitly observe that the constants in inequalities (a_1'') and (a_2') of Corollary 3.1 can be improved when ρ_1 is significantly less than $\Phi(\bar{v})$. For instance, if $n\rho_1 < \Phi(\bar{v}) <$

$\rho_2/2$ with $n \in \mathbb{N}$, arguing in the same way, we can write $\frac{n}{n+1}$ instead of $\frac{1}{2}$ in (a_1'') and $\frac{1}{2}\frac{n}{n+1}$ instead of $\frac{1}{4}$ in (a_1''), while the first endpoint of $\Lambda'_{\rho_1,\rho_2,\bar{v}}$ becomes $\frac{n+1}{n}\frac{\Phi(\bar{v})}{\Upsilon(\bar{v})}$. In particular, if

(a_1''') $\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho[)}\Upsilon(u)}{\rho} < \frac{\Upsilon(\bar{v})}{\Phi(\bar{v})}$ for all ρ small enough;
(a_2'') $\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho_2[)}\Upsilon(u)}{\rho_2} < \frac{1}{2}\frac{\Upsilon(\bar{v})}{\Phi(\bar{v})}$ with $0 < \Phi(\bar{v}) < \rho_2/2$,

then the first endpoint of the interval in the conclusion of Corollary 3.1 becomes $\frac{\Phi(\bar{v})}{\Upsilon(\bar{v})}$.

4. Some applications to elliptic problems

In this section we present some applications of the above results to a variational–hemivariational inequality and to elliptic equations with highly discontinuous nonlinearities.

Let Ω be a non-empty, bounded, open subset of the real Euclidean space $\mathbb{R}^N$, $N \geqslant 3$, with a smooth boundary $\partial\Omega$, and let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega \left|\nabla u(x)\right|^2 dx\right)^{1/2}.$$

Put

$$2^* = \frac{2N}{N-2},$$

this is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subseteq L^p(\Omega)$. It is well known that, if $p \in [1, 2^*]$ then there exists a constant $c_p > 0$ such that

$$\|u\|_{L^p(\Omega)} \leqslant c_p\|u\|$$

for all $u \in H_0^1(\Omega)$, and the embedding is compact whenever $p \in [1, 2^*[$.

Now, let C be a convex closed subset of $H_0^1(\Omega)$ such that $0 \in C$ and let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two functions such that

(l_1) f and g are measurable with respect to each variable separately;
(l_2) there exist $a > 0$, $p \in [1, 2^*[$ such that

$$\max\{\left|f(x,t)\right|; \left|g(x,t)\right|\} \leqslant a\left(1 + |t|^{p-1}\right) \quad \text{in } \Omega \times \mathbb{R}.$$

Now, write

$$F(x,\xi) := \int_0^\xi f(x,t)\,dt, \qquad G(x,\xi) := \int_0^\xi g(x,t)\,dt,$$

if $(x,\xi) \in \Omega \times \mathbb{R}$.

Owing to (l_1) and (l_2), the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$ are well defined and locally Lipschitz in ξ for each fixed $x \in \Omega$. So, we can consider the generalized directional derivatives F° and G° of F and G with respect to the variable ξ.

Now, for λ, $\mu \in \mathbb{R}$, denote by $(P_{\lambda,\mu})$ the following variational–hemivariational inequality problem:

Find $u \in C$ such that

$$-\int_\Omega \nabla u(x)\nabla\big(v(x)-u(x)\big)\,dx \leqslant \lambda \int_\Omega \big[F^\circ\big(x,(u;v-u)\big)+(\mu G)^\circ\big(x,(u;v-u)\big)\big]\,dx$$

for all $v \in C$.

Now, assume also that

(l_3) there exists $0 \leqslant s < 2$ such that, for every $(x,\xi) \in \Omega \times \mathbb{R}$, one has

$$\min\big\{F(x,\xi); -\big|G(x,\xi)\big|\big\} \geqslant -a\big(1+|\xi|^s\big);$$

(l_4) there exists $\gamma > 2$ such that

$$\liminf_{\xi\to 0} \frac{\inf_{x\in\Omega} F(x,\xi)}{|\xi|^\gamma} > -\infty;$$

(l_5) there exists $u_0 \in C$ such that

$$\int_\Omega F\big(x,u_0(x)\big)\,dx < 0 \quad \text{and} \quad \int_\Omega G\big(x,u_0(x)\big)\,dx \leqslant 0.$$

We have the following result.

Theorem 4.1. *Assume that (l_1)–(l_5) hold. Then, for each $\lambda > \frac{\|u_0\|^2}{-\int_\Omega F(x,u_0(x))\,dx}$ there is $\delta > 0$ such that for all $\mu \in [0,\delta]$ the problem $(P_{\lambda,\mu})$ admits at least three solutions.*

Proof. Let us apply Theorem 3.2. For this end choose $X := H_0^1(\Omega)$ and, taking into account (l_5), fix $\lambda > \frac{\|u_0\|^2}{-\int_\Omega F(x,u_0(x))\,dx}$. From (l_4) there are $\eta \in]0,1]$ and $a_1 > 0$ such that

$$\inf_{x\in\Omega} F(x,\xi) \geqslant -a_1|\xi|^\gamma \geqslant -a_1|\xi|^{\gamma^*}$$

for all $\xi \in [-\eta,\eta]$, with $\gamma^* = \gamma$ if $\gamma \leqslant 2^*$ or $\gamma^* = 2^*$ otherwise. So, in view of (l_3) we have

$$F(x,\xi) \geqslant -a_2|\xi|^{\gamma^*}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}$, where $a_2 = \max\{a_1; \sup_{|\xi|>\eta} \frac{a(1+|\xi|^s)}{|\xi|^{\gamma^*}}\}$. Consequently,

$$\int_\Omega F\big(x,u(x)\big)\,dx \geqslant -a_2\|u\|^{\gamma^*}_{L^{\gamma^*}(\Omega)} \geqslant -a_2 c_{\gamma^*}^{\gamma^*}\|u\|^{\gamma^*} := -a_3\|u\|^{\gamma^*}.$$

Since $\gamma^* > 2$ and $\int_\Omega F(x,0)\,dx = 0$, it forces

$$\lim_{r\to 0^+}\left[\frac{1}{r}\inf_{\|u\|\leqslant\sqrt{2r}}\int_\Omega F\big(x,u(x)\big)\,dx\right] = 0.$$

Hence, we can fix $\varepsilon > 0$ such that

$$\frac{\sup_{\|u\|\leqslant\sqrt{2r}}(-\int_\Omega F(x,u(x))\,dx)}{r} + \varepsilon < \frac{1}{\lambda} \tag{4.1}$$

for all positive r small enough.

Now, taking into account that from (l_5) one has $\|u_0\| \neq 0$, fix $r_1 > 0$ such that $\frac{1}{2}\|u_0\|^2 > r_1$ and (4.1) holds, and let $\delta > 0$ such that

$$\delta\left(\frac{\sup_{\|u\|\leqslant\sqrt{2r_1}}(\int_\Omega |G(x,u(x))|\,dx)}{r_1}\right) \leqslant \varepsilon. \tag{4.2}$$

Fix $\mu \in [0,\delta]$ and, for all $u \in X$, define $\Phi(u) := \frac{1}{2}\|u\|^2$, $\Upsilon(u) := -\int_\Omega F(x,u(x))\,dx - \mu\int_\Omega G(x,u(x))\,dx$,

$$j(u) := \begin{cases} 0 & \text{if } u \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

$\Psi(u) := \Upsilon(u) - j(u)$.

From (l_1), (l_2) Υ is a locally Lipschitz functional; while j is clearly convex, proper, and lower semicontinuous. Clearly, Φ, being continuous and convex, is sequentially weakly lower semicontinuous and a standard argument ensures that Υ is sequentially weakly continuous. Moreover, taking (l_3) into account we obtain that for all $u \in C$ one has $\Phi(u) - \lambda\Upsilon(u) + \lambda j(u) \geqslant \frac{1}{2}\|u\|^2 - \lambda a(1+\mu)m(\Omega) - \lambda a(1+\mu)[m(\Omega)]^{\frac{2-s}{2}}\|u\|^s$; therefore $\Phi - \lambda\Psi$ is coercive (hence, it is bounded below). Moreover, it satisfies $(PS)_c$ condition. In fact, let $\{u_n\}$ be a sequence such that $(\Phi - \lambda\Upsilon)(u_n) \to c$, $c \in \mathbb{R}$, and $(\Phi - \lambda\Upsilon)^\circ(u_n; v - u_n) \geqslant -\epsilon_n\|v - u_n\|$ for all $v \in C$, where $\epsilon_n \to 0^+$. Clearly, since $\Phi - \lambda\Upsilon$ is coercive, $\{u_n\}$ is a bounded sequence. Hence, taking a subsequence if necessary, $u_n \rightharpoonup u$ weakly in X and $u_n \to u$ strongly in $L^p(\Omega)$. Then, $u \in C$. So, we have

$$\Phi'(u_n; u - u_n) + \lambda(-\Upsilon)^\circ(u_n; u - u_n) \geqslant -\epsilon_n\|u - u_n\|,$$

$$\langle u_n; u\rangle - \langle u_n, u_n\rangle + \lambda(-\Upsilon)^\circ(u_n; u - u_n) \geqslant -\epsilon_n\|u - u_n\|,$$

$$\|u_n\|^2 - \epsilon_n\|u - u_n\| \leqslant \|u_n\|\|u\| + \lambda(-\Upsilon)^\circ(u_n; u - u_n).$$

Taking into account that the functional Υ is actually defined and locally Lipschitz in $L^p(\Omega)$ and that one has $(-\Upsilon|_X)^\circ(u; v) \leqslant [(-\Upsilon)^\circ]|_X(u; v)$ for all $u, v \in X$ (see, for instance, [16, p. 111]), the upper semicontinuity of $(-\Upsilon)^\circ$ in the strong topology of $L^p(\Omega) \times L^p(\Omega)$ (see, for instance, [29, Proposition 1.1]) then implies $\limsup_{n\to\infty}(-\Upsilon)^\circ(u_n; u - u_n) \leqslant 0$ and, therefore, the previous inequality ensures that $\limsup_{n\to+\infty}\|u_n\| \leqslant \|u\|$. Hence, since X is uniformly convex, from

[15, Proposition III.30] one has that $u_n \to u$ strongly in X and our claim is proved. Therefore, (b_1) of Theorem 3.2 is verified.

Now, taking into account (4.2), one has

$$\begin{aligned}\varphi^{(1)}(r_1) &\leqslant \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} = \frac{\chi(r_1)}{r_1}\\ &\leqslant \frac{\sup_{\|u\|\leqslant\sqrt{2r_1}}(-\int_\Omega F(x,u(x))\,dx)}{r_1}\\ &\quad + \delta\left(\frac{\sup_{\|u\|\leqslant\sqrt{2r_1}}(\int_\Omega |G(x,u(x))|\,dx)}{r_1}\right)\\ &\leqslant \frac{\sup_{\|u\|\leqslant\sqrt{2r_1}}(-\int_\Omega F(x,u(x))\,dx)}{r_1} + \varepsilon < \frac{1}{\lambda}.\end{aligned}$$

On the other hand, also taking Remark 3.11 into account, one has

$$\begin{aligned}\varphi^{(2)}(r_1) &\geqslant \frac{1}{2}\frac{\Psi(u_0)}{\Phi(u_0)} = \frac{-\int_\Omega F(x,u_0(x))\,dx - \mu\int_\Omega G(x,u_0(x))\,dx}{\|u_0\|^2}\\ &\geqslant \frac{-\int_\Omega F(x,u_0(x))\,dx}{\|u_0\|^2} > \frac{1}{\lambda}.\end{aligned}$$

Hence, one has

$$\varphi^{(1)}(r_1) \leqslant \frac{\chi(r_1)}{r_1} < \frac{1}{\lambda} < \frac{1}{2}\frac{\Psi(u_0)}{\Phi(u_0)} \leqslant \varphi^{(2)}(r_1).$$

Therefore, the assumption (a_1') of Theorem 3.2 is satisfied. So, owing to Theorem 3.2 there are $u_i \in C$, $i = 1, 2, 3$, such that

$$(\Phi - \lambda\Upsilon)^\circ(u_i; v - u_i) \geqslant 0$$

for all $v \in C$. Hence,

$$\begin{aligned}(\Phi)^\circ(u_i; v-u_i) + \lambda(-\Upsilon)^\circ(u_i; v-u_i) &\geqslant (\Phi-\lambda\Upsilon)^\circ(u_i; v-u_i) \geqslant 0,\\ \int_\Omega \nabla u(x)\nabla\big(v(x)-u(x)\big)\,dx &\geqslant -\lambda(-\Upsilon)^\circ(u_i; v-u_i),\\ -\int_\Omega \nabla u(x)\nabla\big(v(x)-u(x)\big)\,dx &\leqslant \lambda(-\Upsilon)^\circ(u_i; v-u_i).\end{aligned}$$

So, taking into account that from [16, Theorem 2.7.5] one has

$$(-\Upsilon)^\circ(u_i; v-u_i) \leqslant \int_\Omega (F+\mu G)^\circ\big(x, (u_i; v-u_i)\big)\,dx,$$

we obtain

$$-\int_{\Omega} \nabla u_i(x)\nabla\big(v(x)-u_i(x)\big)\,dx \leqslant \lambda \int_{\Omega}\big[F^{\circ}\big(x,(u;v-u_i)\big)+\mu G^{\circ}\big(x,(u_i;v-u_i)\big)\big]\,dx$$

for all $v \in C$. $\square$

Remark 4.1. With respect to [29, Theorem 4.1], in Theorem 4.1, in addition, the following condition is assumed,

$$\int_{\Omega} G\big(x,u_0(x)\big)\,dx \leqslant 0.$$

Clearly, if such a condition is not satisfied, we obtain the same conclusion for all $\mu \in [-\delta, 0]$.

We explicitly observe that the conclusions of Theorem 4.1 and [29, Theorem 4.1] are different. In particular, Theorem 4.1 ensures three solutions to $(P_{\lambda,\mu})$ for all $\lambda > \lambda^*$, where λ^* is precisely determined, and for all μ small enough. On the contrary, [29, Theorem 4.1] ensures three solutions to $(P_{\lambda,\mu})$ for all μ small enough and for all $\lambda \in \Lambda_\mu$, where Λ_μ is an interval which is not precisely localized.

We also point out that, if the condition

$$\big|G(x,\xi)\big| \leqslant a\big(1+|\xi|^s\big),$$

$(x,\xi) \in \Omega \times \mathbb{R}$, $0 \leqslant s < 2$, $a > 0$, is not satisfied then, arguing as in the proof of Theorem 4.1 and applying Theorem 3.1 instead of Theorem 3.2, we obtain the same conclusion with "two solutions" instead of "three solutions." To be precise, as seen in the proof of Theorem 4.1, we obtain

$$\lim_{r\to 0^+}\left[\frac{1}{r}\inf_{\|u\|\leqslant\sqrt{2r}}\int_{\Omega} F\big(x,u(x)\big)\,dx\right]=0$$

and, in a similar way, from $F(x,\xi) \geqslant -a(1+|\xi|^s)$, $(x,\xi) \in \Omega \times \mathbb{R}$, $0 \leqslant s < 2$, $a > 0$, we obtain

$$\lim_{r\to +\infty}\left[\frac{1}{r}\inf_{\|u\|\leqslant\sqrt{2r}}\int_{\Omega} F\big(x,u(x)\big)\,dx\right]=0.$$

So, there are r_1, r_2, with $0 < r_1 < \Phi(u_0) < r_2$, such that

$$\frac{\sup_{\|u\|\leqslant\sqrt{2r_1}}(-\int_{\Omega} F(x,u(x))\,dx)}{r_1}+\varepsilon < \frac{1}{\lambda}$$

and

$$\frac{\sup_{\|u\|\leqslant\sqrt{2r_2}}(-\int_{\Omega} F(x,u(x))\,dx)}{r_2}+\varepsilon < \frac{1}{\lambda}.$$

Fixed $\delta > 0$ such that

$$\delta\left(\frac{\sup_{\|u\|\leqslant\sqrt{2r_1}}(\int_\Omega |G(x,u(x))|\,dx)}{r_1}\right)\leqslant\varepsilon \quad\text{and}\quad \delta\left(\frac{\sup_{\|u\|\leqslant\sqrt{2r_2}}(\int_\Omega |G(x,u(x))|\,dx)}{r_2}\right)\leqslant\varepsilon,$$

we obtain $\varphi^{(1)}(r_1) < \frac{1}{\lambda}$ and $\varphi^{(1)}(r_2) < \frac{1}{\lambda}$ for all $\mu \in [0,\delta]$. On the other hand, arguing as in the proof of Theorem 4.1, we obtain $\varphi_2(r_1,r_2) \geqslant \frac{1}{2}\frac{\Psi(u_0)}{\Phi(u_0)} > \frac{1}{\lambda}$. Hence, the assumption (a) of Theorem 3.1 is verified and our claim is proved.

When $C := H_0^1(\Omega)$ and f and g are continuous, Theorem 4.1 ensures three weak solutions to

$$\begin{cases} \Delta u = \lambda\big(f(u)+\mu g(u)\big) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{$AD_{\lambda,\mu}$}$$

With respect to [36, Theorem 4], the same previous remarks hold. However, in the case of Dirichlet problems, we can obtain more precise results applying Theorem 3.3 instead of Theorem 3.2, as we will see below.

Write, for $h : \Omega\times\mathbb{R}\to\mathbb{R}$ and $(x,z)\in\Omega\times\mathbb{R}$,

$$h^-(x,z) = \lim_{\delta\to 0^+}\operatorname*{ess\,inf}_{|\zeta-z|<\delta} h(x,\zeta), \qquad h^+(x,z) = \lim_{\delta\to 0^+}\operatorname*{ess\,sup}_{|\zeta-z|<\delta} h(x,\zeta).$$

Moreover, denote by $\mathcal{H}$ the family of locally bounded functions $h : \Omega\times\mathbb{R}\to\mathbb{R}$ such that

(m_1) $x \to h(x,z)$ is measurable for all $z\in\mathbb{R}$;

(m_2) there exists a set $\Omega_h \subseteq \Omega$ with $m(\Omega_h)=0$ such that the set

$$D_h := \bigcup_{x\in\Omega\setminus\Omega_h}\big\{z\in\mathbb{R}\ \big|\ h(x,\cdot)\text{ is discontinuous at } z\big\}$$

has measure zero.

We recall that a function $h : \Omega\times\mathbb{R}\to\mathbb{R}$ is called superpositionally measurable when $x\to h(x,u(x))$ is measurable for all measurable $u:\Omega\to\mathbb{R}$.

Finally, put $\delta(x) = \sup\{\delta\in\mathbb{R}^+:\ B(x,\delta)\subseteq\Omega\}$ for all $x\in\Omega$, and

$$D = \sup_{x\in\Omega}\delta(x). \tag{4.3}$$

Simple calculations show that there is $x_0\in\Omega$ such that $B(x_0,D)\subseteq\Omega$.

We have the following result.

Theorem 4.2. *Let $f, g\in\mathcal{H}$ be such that (l_2) and (l_4) hold. Assume also that f and g are non-positive functions and*

(l_3') *there exists $0\leqslant s<2$ such that, for every $(x,\xi)\in\Omega\times\mathbb{R}$, one has*

$$F(x,\xi)\geqslant -a\big(1+|\xi|^s\big);$$

(l'_5) *there exists $d > 0$ such that*

$$\sup_{x\in\Omega} F(x,d) < 0.$$

Further, assume that, for all $\mu \geqslant 0$ small enough, one has

(m_3) *the functions $(f+\mu g)^-$ and $(f+\mu g)^+$ are superpositionally measurable*;
(m_4) *for almost every $x \in \Omega$ and each $z \in D_f \cup D_g$ such that $(f+\mu g)^-(x,z) \leqslant 0 \leqslant (f+\mu g)^+(x,z)$ one has $(f+\mu g)(x,z) = 0$.*

Then, for each $\lambda > \frac{2^2(2^N-1)}{D^2}\frac{d^2}{[-\sup_{x\in\Omega} F(x,d)]}$ there is $\delta > 0$ such that for all $\mu \in [0,\delta]$ the problem

$$\begin{cases} \Delta u = \lambda\big(f(x,u) + \mu g(x,u)\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$D_{\lambda,\mu}$}$$

admits at least three non-negative weak solutions.

Proof. Let us apply Theorem 3.3 in the form of Corollary 3.1. To this end choose $X := H_0^1(\Omega)$ and fix λ as in the conclusion. Arguing as in the proof of Theorem 4.1, we obtain

$$\lim_{r\to 0^+}\left[\frac{1}{r}\inf_{\|u\|\leqslant\sqrt{2r}}\int_\Omega F\big(x,u(x)\big)\,dx\right] = 0,$$

and, in a similar way, from (l'_3) we also obtain

$$\lim_{r\to +\infty}\left[\frac{1}{r}\inf_{\|u\|\leqslant\sqrt{2r}}\int_\Omega F\big(x,u(x)\big)\,dx\right] = 0.$$

Therefore,

$$\frac{\sup_{\|u\|\leqslant\sqrt{2\rho}}(-\int_\Omega F(x,u(x))\,dx)}{\rho} + \varepsilon < \frac{1}{\lambda} \tag{4.4}$$

for all positive ρ small enough, and

$$\frac{\sup_{\|u\|\leqslant\sqrt{2\rho}}(-\int_\Omega F(x,u(x))\,dx)}{\rho} + \varepsilon < \frac{1}{2\lambda} \tag{4.5}$$

for all positive ρ large enough. Let $\delta > 0$ such that

$$\delta\left(\frac{\sup_{\|u\|\leqslant\sqrt{2\rho}}(\int_\Omega |G(x,u(x))|\,dx)}{\rho}\right) \leqslant \varepsilon \tag{4.6}$$

for all positive ρ small enough, and

$$\delta\left(\frac{\sup_{\|u\|\leqslant\sqrt{2\rho}}(\int_\Omega |G(x,u(x))|\,dx)}{\rho}\right)\leqslant\varepsilon \tag{4.7}$$

for all positive ρ large enough.

Now, fix $\mu\in[0,\delta]$ and put, for all $u\in X$, $\Phi(u):=\frac{1}{2}\|u\|^2$ and

$$\Upsilon(u):=-\int_\Omega F\big(x,u(x)\big)\,dx-\mu\int_\Omega G\big(x,u(x)\big)\,dx.$$

From (l_1), (l_2) Υ is a locally Lipschitz functional. Moreover, standard arguments ensure that it is sequentially weakly continuous and Φ is convex and sequentially weakly lower semicontinuous.

Let $M>0$, we claim that $\Phi-\lambda\Upsilon_M$ satisfies $(PS)_c$ condition. Let $\{u_n\}$ be a sequence such that $(\Phi-\lambda\Upsilon_M)(u_n)\to c$, $c\in\mathbb{R}$, and $(\Phi-\lambda\Upsilon_M)^\circ(u_n;v-u_n)\geqslant-\epsilon_n\|v-u_n\|$ for all $v\in X$, where $\epsilon_n\to0^+$. Clearly, since Φ is coercive and hence $\Phi-\lambda\Upsilon_M$ is also coercive, $\{u_n\}$ is a bounded sequence. Hence, taking a subsequence if necessary, $u_n\rightharpoonup u$ weakly in X and $u_n\to u$ strongly in $L^p(\Omega)$. So, we have

$$\Phi'(u_n;u-u_n)+\lambda(-\Upsilon_M)^\circ(u_n;u-u_n)\geqslant-\epsilon_n\|u-u_n\|,$$
$$\langle u_n;u\rangle-\langle u_n,u_n\rangle+\lambda(-\Upsilon_M)^\circ(u_n;u-u_n)\geqslant-\epsilon_n\|u-u_n\|,$$
$$\|u_n\|^2-\epsilon_n\|u-u_n\|\leqslant\|u_n\|\|u\|+\lambda(-\Upsilon_M)^\circ(u_n;u-u_n).$$

The upper semicontinuity of $(-\Upsilon_M)^\circ$, arguing as in the proof of Theorem 4.1, then implies $\limsup_{n\to+\infty}\|u_n\|\leqslant\|u\|$. Hence, from [15, Proposition III.30] one has that $u_n\to u$ strongly in X and our claim is proved, namely, (b_3) of Corollary 3.1 is verified.

Now, from (4.4)–(4.7) one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho[)}\Upsilon(u)}{\rho}<\frac{1}{\lambda} \tag{4.8}$$

for all positive ρ small enough, and

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,\rho[)}\Upsilon(u)}{\rho}<\frac{1}{2\lambda} \tag{4.9}$$

for all positive ρ large enough.

Next, put

$$u_d(x):=\begin{cases}0 & \text{if } x\in\Omega\setminus B(x_0,D),\\ \frac{2d}{D}\Big[D-\sqrt{\sum_{i=1}^N\big(x_i-x_i^0\big)^2}\,\Big] & \text{if } x\in B(x_0,D)\setminus B(x_0,D/2),\\ d & \text{if } x\in B(x_0,D/2).\end{cases}$$

We have

$$\Phi(u_d) = \frac{1}{2}\|u_d\|^2 = \frac{1}{2}\int_\Omega \left|\nabla u_d(x)\right|^2 dx = \frac{1}{2}\int_{B(x_0,D)\setminus B(x_0,D/2)} \frac{(2d)^2}{D^2}\,dx$$
$$= \frac{1}{2}\frac{(2d)^2}{D^2}\big(m\big(B(x_0,D)\big) - m\big(B(x_0,D/2)\big)\big) = \frac{1}{2}\frac{(2d)^2}{D^2}\frac{\pi^{N/2}}{\Gamma(1+N/2)}\big(D^N - (D/2)^N\big).$$

Moreover, we obtain

$$\Upsilon(u_d) \geqslant \int_{B(x_0,D/2)} \big[-F(x,d)\big]\,dx \geqslant -\sup_{x\in\Omega} F(x,d)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\frac{D^N}{2^N}.$$

Hence, one has

$$\frac{1}{2}\frac{\Upsilon(u_d)}{\Phi(u_d)} \geqslant \frac{D^2}{2^2(2^N-1)}\frac{[-\sup_{x\in\Omega} F(x,d)]}{d^2} > \frac{1}{\lambda}. \tag{4.10}$$

Therefore, by choosing $\rho_1 < \Phi(u_d)$ such that (4.8) holds, and by choosing $\rho_2 > 2\Phi(u_d)$ such that (4.9) holds, owing to (4.10), conditions (a_1'') and (a_2') of Corollary 3.1 are verified.

Now, we claim that the generalized critical points of $\Phi - \lambda\Upsilon$ are weak solutions for the problem $(D_{\lambda,\mu})$. To this end, let $u_0 \in H_0^1(\Omega)$ such that

$$(\Phi - \lambda\Upsilon)^\circ(u_0; v - u_0) \geqslant 0 \tag{4.11}$$

for all $v \in H_0^1(\Omega)$.

From (4.11) we obtain

$$-\int_\Omega \nabla u_0(x)\nabla v(x)\,dx \leqslant \lambda(-\Upsilon)^\circ(u_0, v) \tag{4.12}$$

for all $v \in H_0^1(\Omega)$. Clearly, setting $L(v) = -\int_\Omega \nabla u_0(x)\nabla v(x)\,dx$, L is a continuous and linear functional on $H_0^1(\Omega)$; for which, (4.12) signifies $L \in \lambda\partial(-\Upsilon)(u_0)$. Now, since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, from [16, Theorem 2.2] we obtain $\partial(-\Upsilon)|_{H_0^1(\Omega)}(u_0) \subseteq \partial(-\Upsilon)|_{L^2(\Omega)}(u_0)$, so that L is a continuous and linear functional on $L^2(\Omega)$. Therefore, there exists $w \in L^2(\Omega)$ such that $L(v) = \int_\Omega w(x)v(x)\,dx$ for all $v \in L^2(\Omega)$. From [21, Theorem 9.15, p. 241] there is a unique $\bar{u} \in W^{2,2} \cap H_0^1(\Omega)$ such that $\Delta\bar{u} = w$. In particular, we have

$$\int_\Omega \Delta\bar{u}(x)v(x)\,dx = -\int_\Omega \nabla\bar{u}(x)\nabla v(x)\,dx$$

for all $v \in H_0^1(\Omega)$. Hence, $-\int_\Omega \nabla u_0(x)\nabla v(x)\,dx = L(v) = \int_\Omega w(x)v(x)\,dx = \int_\Omega \Delta\bar{u}(x)\cdot v(x)\,dx = -\int_\Omega \nabla\bar{u}(x)\nabla v(x)\,dx$ and since a continuous and linear functional L on $H_0^1(\Omega)$ is uniquely determined by a function in $H_0^1(\Omega)$ (see [25, Theorem 5.9.3, p. 295]), we have $\bar{u} = u_0$; so that, $u_0 \in W^{2,2}(\Omega)$ and

$$\int_{\Omega} \Delta u_0(x) v(x)\,dx = -\int_{\Omega} \nabla u_0(x) \nabla v(x)\,dx \tag{4.13}$$

for all $v \in H_0^1(\Omega)$.

From (4.12), (4.13) and [16, Corollary, p. 111] we have

$$\Delta u_0(x) \in \lambda\big[(f+\mu g)^-\big(x, u_0(x)\big), (f+\mu g)^+\big(x, u_0(x)\big)\big] \tag{4.14}$$

for almost everywhere $x \in \Omega$.

Clearly, for almost every $x \in \Omega \setminus u_0^{-1}(D_f \cup D_g)$ condition (4.14) reduces to

$$\Delta u_0(x) = \lambda(f+\mu g)\big(x, u_0(x)\big),$$

while for almost every $x \in u_0^{-1}(D_f \cup D_g)$, since $m(D_f \cup D_g) = 0$, from Lemma 1 of [18] we obtain $\Delta u_0(x) = 0$ and, from (m_4) and (4.14), we obtain $\lambda(f+\mu g)(x, u_0(x)) = 0$ for almost every $x \in u_i^{-1}(D_f \cup D_g)$. That is

$$-\Delta u_0(x) = 0 = \lambda(f+\mu g)\big(x, u_0(x)\big)$$

for almost all $x \in u_0^{-1}(D_f \cup D_g)$. Hence, our claim is proved.

Finally, we verify that $\Phi - \lambda\Upsilon$ satisfies the assumption $(3')$ of Corollary 3.1 (see Remark 3.9). Let u_1 and u_2 be two local minima for $\Phi - \lambda\Upsilon$. Then u_1 and u_2 are generalized critical points for $\Phi - \lambda\Upsilon$ and, hence, they are two weak solutions for the problem $(D_{\lambda,\mu})$. Since f and g are non-positive functions, from the Weak Maximum Principle (see [21, Theorem 8.1]) we obtain $u_1(x) \geqslant 0$, $u_2(x) \geqslant 0$ for all $x \in \Omega$. Therefore, one has $tu_1(x) + (1-t)u_2(x) \geqslant 0$, $-F(x, tu_1(x) + (1-t)u_2(x)) - \mu G(x, tu_1(x) + (1-t)u_2(x)) \geqslant 0$, $x \in \Omega$, $t \in [0,1]$, and, hence, $\Upsilon(tu_1 + (1-t)u_2) \geqslant 0$ for all $t \in [0,1]$.

Since all the assumptions of Corollary 3.1 are satisfied, the functional $\Phi - \lambda\Upsilon$ admits at least three generalized critical points which are, as seen before, weak solutions for $(D_{\lambda,\mu})$ and they are non-negative functions; hence, the conclusion is achieved. □

When f and g are not depending on $x \in \Omega$, Theorem 4.2 takes simpler forms. By way of example, we point out the following result.

Theorem 4.3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two locally bounded, and almost everywhere continuous functions such that (l_2) holds. Assume that g is negative in $\mathbb{R}$ and f is non-positive and non-zero in $]0, +\infty[$ such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t^\beta} = \lim_{t\to +\infty} \frac{f(t)}{t^\alpha} = 0 \quad \textit{for some } \alpha \textit{ and } \beta \textit{ such that } 0 \leqslant \alpha < 1 < \beta.$$

Then, for each $\lambda > \frac{2^2(2^N-1)}{D^2} \inf_{d>0} \frac{d^2}{[-\int_0^d f(t)\,dt]}$, there is $\delta > 0$ such that for all $\mu \in]0, \delta]$ the problem

$$\begin{cases} \Delta u = \lambda\big(f(u) + \mu g(u)\big) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{$AD_{\lambda\mu}$}$$

admits at least three positive weak solutions.

Proof. Without loss of generality, we can assume $f(t)=0$ for all $t \leqslant 0$. Since f, g are locally bounded and continuous almost everywhere, therefore (m_1), (m_2) and (m_3) are verified. From $f(t)+\mu g(t)<0$ for all $t \in \mathbb{R}$ we obtain (m_4). Moreover, $\lim_{t\to 0^+} \frac{f(t)}{t^\beta}=0$ $(\beta>1)$ implies (l_4), while $\lim_{t\to+\infty} \frac{f(t)}{t^\alpha}=0$ $(0 \leqslant \alpha<1)$ implies (l_3'). Hence, Theorem 4.2 and the Strong Maximum Principle (see, for instance, [21, Theorem 8.19]) ensure the conclusion. □

Remark 4.2. Clearly, in Theorem 4.3 it is enough to assume that g is non-positive in $\mathbb{R}$ provided that $f(t)+\mu g(t)<0$ for all $t \in D_f \cup D_g$ and $g(0) \neq 0$.

Remark 4.3. One of the key assumptions in [29, Theorem 4.2] is

$$\bigl|G(x,\xi)\bigr| \leqslant a\bigl(1+|\xi|^s\bigr) \tag{4.15}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}$ and for some $0 \leqslant s<2$ and $a>0$.

We explicitly observe that, in Theorems 4.2 and 4.3, condition (4.15) is not requested. Hence, when (4.15) is not satisfied, we can apply our results. The next example is in this direction.

Example 4.1. Let $\Omega=\{x \in \mathbb{R}^3\colon\ |x| \leqslant 1\}$. Let f and g be two functions defined as follows

$$f(u):=\begin{cases} 0 & \text{if } u \leqslant 1,\\ \sqrt{u} & \text{if } u>1,\end{cases}$$

$$g(u):=|u|^3+1$$

for all $u \in \mathbb{R}$.

Owing to Theorem 4.3, for each $\lambda>56\sqrt[3]{4}$ there is $\delta>0$ such that for all $\mu \in\,]0,\delta]$ the problem

$$\begin{cases} -\Delta u=\lambda\bigl(f(u)+\mu g(u)\bigr) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{P}$$

admits at least three positive weak solutions.

Clearly, we cannot apply [29, Theorem 4.2] since (4.11) is not satisfied.

Remark 4.4. We point out that the solutions given by Theorem 4.2 (and, hence, by Theorem 4.3), are actually weak solutions; on the contrary, in most of the papers that investigate Dirichlet problems for elliptic equations with discontinuous nonlinearities, the solutions are multi-valued solutions, namely solutions for the corresponding differential inclusion obtained by filling the gaps at the discontinuity points (see, for instance, [20,22] and the references therein). This is due to the assumption (m_4) that allows us to apply a classical lemma of [18]. We recall that in the case of Dirichlet problems for elliptic equations having discontinuous nonlinear terms, the assumption (m_4) was studied and developed in [28] and [14], where the approach taken was entirely based on set-valued analysis arguments.

We also observe that, in Theorem 4.2, the set of discontinuity points may be also uncountable as the following easy example shows.

Example 4.2. Let Ω be a bounded domain in $\mathbb{R}^N$. Let f and g be two functions defined as follows

$$f(u) := \begin{cases} u^2 & \text{if } u \leqslant 1, \\ 0 & \text{if } u > 1, \end{cases}$$

$$g(u) := \begin{cases} 1 & \text{if } u \in C, \\ 0 & \text{otherwise,} \end{cases}$$

for all $u \in \mathbb{R}$, where C denotes the Cantor "middle third" set. We observe that the function $\lambda(f + \mu g)$ ($\lambda > 0$, $\mu > 0$) has an uncountable set of discontinuity points. Nevertheless, owing to Theorem 4.3 (see Remark 4.2), for all $\lambda > 12\frac{2^N - 1}{D^2}$ there is $\delta > 0$ such that for all $\mu \in]0, \delta]$ problem (P) admits at least three positive weak solutions.

Remark 4.5. We explicitly observe that Theorems 4.2 and 4.3 are novel results also for continuous functions. In particular, when f and g are non-positive functions, Theorem 4.2 improves [36, Theorem 4] as Example 4.3 shows. On the other hand, in [36, Theorem 4] no sign assumption on f and g is made; in this case, we refer to Remark 4.1.

Again in the continuous case, we can also compare our results with a classical theorem established in [34] (see also [1]). To be precise, let $l : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the structure assumption

(l_2) there exist $a > 0$, $p \in [1, 2^*[$ such that

$$\left|l(t)\right| \leqslant a\left(1 + |t|^{p-1}\right) \quad \text{in } \Omega \times \mathbb{R},$$

and such that

(p_1) $\lim_{t \to 0} \frac{l(t)}{|t|} = 0$;
(p_2) there are constants $\gamma > 2$ and $r \geqslant 0$ such that for $|t| \geqslant r$,

$$0 < \gamma \int_0^t l(\xi)\, d\xi \leqslant t l(t)$$

(hence, l is superlinear at infinity).

Therefore, owing to [34, Theorem 2.15] the problem

$$\begin{cases} -\Delta u = l(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

admits at least one non-zero weak solution.

We note that, if $l \equiv f$ (and $g \equiv 0$), the assumption (p_2) of [34, Theorem 2.15] and the assumption (l_4) of Theorem 4.2 are opposite conditions. Nevertheless, if we choose

$$l \equiv \lambda(f + \mu g)$$

for suitable f, g, λ and μ, the conditions (p_1) and (p_2) may, or may not, be satisfied, while Theorem 4.2 can ensure three weak solutions, as Example 4.3 shows.

Finally, about the continuous case, we also cite the recent papers [26] and [27], where multiple solutions for problem (P_1) are ensured. It is easy to verify that the results in these works are mutually independent with Theorem 4.2. For instance, in [26, Theorem 1], the authors establish multiple solutions for problem (P_1) when

$$l \equiv f + \mu g,$$

where f is, in particular, an odd function. In the next example, problem (P_1) admits three weak solutions owing to Theorem 4.2, while [26, Theorem 1] cannot be applied.

Example 4.3. Let Ω be a bounded domain in $\mathbb{R}^N$. Let f and g be two functions defined as follows

$$f(u) := u^2 e^{-u},$$
$$g(u) := |u| + 1$$

for all $u \in \mathbb{R}$. Put

$$l_\mu(u) = f(u) + \mu g(u)$$

for all $u \in \mathbb{R}$ and $\mu > 0$. By choosing $d > 0$ such that $\frac{2^2(2^N-1)}{D^2} \frac{d^2}{[\int_0^d t^2 e^{-t}\,dt]} < 1$, owing to Theorem 4.3, for each positive real μ small enough, the problem

$$\begin{cases} -\Delta u = l_\mu(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_2}$$

admits at least three positive weak solutions. Clearly, we cannot apply [34, Theorem 2.15] to l_μ, since (p_1) and (p_2) are not satisfied. Moreover, if we apply [36, Theorem 4], we obtain that, for some $\bar{\lambda} > 0$ (and for μ small enough), the problem

$$\begin{cases} -\Delta u = \bar{\lambda} l_\mu(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_3}$$

admits three weak solutions. We explicitly observe that [36, Theorem 4] ensures no estimate on the value of $\bar{\lambda}$.

We also observe that, by choosing

$$l_\mu(u) = f(u) + \mu g_1(u) = u^2 e^{-u} + \mu\big(|u|^3 + 1\big)$$

for all $u \in \mathbb{R}$, again owing to Theorem 4.3, the problem (P_2), for each positive real number μ small enough, admits three positive weak solutions. In this case, we cannot apply [36, Theorem 4] since l_μ is superlinear and we cannot apply [34, Theorem 2.15] since (p_1) is not satisfied. Moreover, if

$$l_\mu(u) = f(u) + \mu g_2(u) = u^2 e^{-u} + \mu\big(|u|^3\big)$$

for all $u \in \mathbb{R}$, our result ensures two positive weak solutions for problem (P_2) (with μ small enough), while, as before, we cannot apply [36, Theorem 4], and [34, Theorem 2.15] ensures only one non-zero weak solution.

Finally, we remark that we cannot apply [26, Theorem 1] to l_μ in all previous cases since f is not an odd function.

We conclude this section investigating elliptic problems involving the p-Laplacian, with $p > N$. We observe that, in this case, the results in Section 3 can be fully applied because $W_0^{1,p}$ is embedded in C^0 and so we can estimate $\chi(r)$ in an optimum way (see Remark 3.11). For simplicity, we consider the following autonomous Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and λ is a positive real parameter.

As usual, Ω is a non-empty bounded open subset of the real Euclidean space $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$, and $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega \left|\nabla u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Put

$$k := \sup\left\{\frac{\max_{x\in\overline{\Omega}} |u(x)|}{\|u\|}\colon\ u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}.$$

Since $p > N$, one has $k < +\infty$. In addition, it is known ([40, formula (6b)]) that

$$k \leqslant \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}}\left[\Gamma\left(1+\frac{N}{2}\right)\right]^{\frac{1}{N}}\left(\frac{p-1}{p-N}\right)^{1-\frac{1}{p}}\left[m(\Omega)\right]^{\frac{1}{N}-\frac{1}{p}},$$

where Γ denotes the Gamma function and $m(\Omega)$ is the Lebesgue measure of Ω, and equality occurs when Ω is a ball.

Put

$$K := \left[\frac{2^{(p-N)}(2^N-1)\pi^{N/2}}{D^{(p-N)}\Gamma(1+N/2)}\right]^{1/p} k, \tag{4.16}$$

$$R := \frac{\pi^{N/2}D^N}{\Gamma(1+N/2)2^N}\left(\frac{1}{K}\right)^p\frac{1}{m(\Omega)} = \frac{D^p}{2^p(2^N-1)}\left(\frac{1}{k}\right)^p\frac{1}{m(\Omega)}, \tag{4.17}$$

where D is given by (4.3).

Here, and in the sequel, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous almost everywhere function, namely $m(D_f) = 0$, where $D_f = \{z \in \mathbb{R}\colon\ f \text{ is discontinuous at } z\}$. Moreover, we also assume that

(a) for each $\rho > 0$ there is a constant M_ρ such that

$$\sup_{|z|\leqslant\rho} \left|f(z)\right| \leqslant M_\rho.$$

Theorem 4.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative, continuous almost everywhere function such that (a) holds. Put $F(s) := \int_0^s f(t)\,dt$ for all $s\in\mathbb{R}$ and assume*:

(h_1) *there exist three positive constants c_1, d and c_2, with $c_1 < Kd < (\frac{1}{2})^{\frac{1}{p}} c_2$, such that*

$$\frac{F(c_1)}{c_1^p} < \frac{R}{2}\frac{F(d)}{d^p},$$
$$\frac{F(c_2)}{c_2^p} < \frac{R}{4}\frac{F(d)}{d^p};$$

(h_2) *for all $z\in D_f$ the condition $f^-(z)=0$ implies $f(z)=0$.*

Then, for each $\lambda\in\Lambda$, where

$$\Lambda = \left]\frac{2^{p+1}(2^N-1)}{pD^p}\frac{d^p}{F(d)}; \min\left\{\frac{1}{m(\Omega)pk^p}\frac{c_1^p}{F(c_1)}; \frac{1}{2m(\Omega)pk^p}\frac{c_2^p}{F(c_2)}\right\}\right[,$$

the problem (D_λ) possesses at least three non-negative weak solutions u_i, $i=1,2,3$, in $W_0^{1,p}(\Omega)$ such that $\max_{x\in\Omega}|u_i(x)| < c_2$.

Proof. Let X be the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm $\|u\| = (\int_\Omega |\nabla u(x)|^p\,dx)^{\frac{1}{p}}$. For each $u\in X$, put $\Phi(u) := \frac{1}{p}\|u\|^p$ and $\Upsilon(u) := \int_\Omega F(u(x))\,dx$. Our goal is to apply Corollary 3.1 to Φ and Υ. From standard results, Φ and Υ are locally Lipschitz, Φ is convex and weakly sequentially lower semicontinuous, and Υ, since f satisfies (a) and X is compactly embedded in $C(\overline{\Omega})$, is weakly sequentially continuous.

Now, fixed $\lambda>0$, we claim that the functional $\Phi-\lambda\Upsilon$ satisfies $(PS)_c^M$-condition for all $M>0$. To this end, let $\{u_n\}$ be a sequence in X such that $\Phi(u_n)-\lambda\Upsilon_M(u_n)\to c\in\mathbb{R}$ and $(\Phi-\lambda\Upsilon_M)^\circ(u_n; v-u_n) \geqslant -\epsilon_n\|v-u_n\|$ for all $v\in X$, where $\epsilon_n\to 0^+$. Since Φ is coercive and hence $\Phi-\lambda\Upsilon_M$ is also coercive, the sequence $\{u_n\}$ is bounded. Hence, taking a subsequence if necessary, $u_n \rightharpoonup u$ in X and $u_n\to u$ in $L^p(\Omega)$. From the above expression, written with $v=u$, we infer

$$\Phi'(u_n)(u-u_n) + \lambda(-\Upsilon_M)^\circ(u_n; u-u_n) \geqslant -\epsilon_n\|u-u_n\|$$

for all $n\in\mathbb{N}$.

Clearly, $\Phi'(u_n)(u-u_n) = \int_\Omega [|\nabla u_n(x)|^{p-2}\nabla u_n(x)\nabla(u(x)-u_n(x))]\,dx = \int_\Omega [|\nabla u_n(x)|^{p-2}\cdot \nabla u_n(x)\nabla u(x)]\,dx - \|u_n\|^p$ and, bearing in mind that

$$|a|^{p-1}|b| \leqslant \frac{p-1}{p}|a|^p + \frac{|b|^p}{p} \quad \text{for all } a,b\in\mathbb{R},$$

one has

$$\Phi'(u_n)(u-u_n) \leqslant \frac{\|u\|^p}{p} - \frac{\|u_n\|^p}{p}$$

for each $n \in \mathbb{N}$. So, we obtain

$$-\epsilon_n\|u-u_n\| + \frac{\|u_n\|^p}{p} \leqslant \lambda(-\Upsilon_M)^\circ(u_n; u-u_n) + \frac{\|u\|^p}{p}.$$

From this, taking the upper semicontinuity of $(-\Upsilon_M)^\circ$ into account (see the proof of Theorem 4.1 for more details), we obtain

$$\limsup_{n\to+\infty}\|u_n\| \leqslant \|u\|.$$

Thus, since X is uniformly convex, Proposition III.30 of [15] ensures that $\{u_n\}$ converges to u in X. Hence, our claim is proved.

At this point, we show that Φ and Υ satisfy (a_1'') and (a_2') in Corollary 3.1. Let $x_0 \in \Omega$ be such that $B(x_0, D) \subseteq \Omega$. Put

$$u_d(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, D), \\ \frac{2d}{D}\big[D - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}\,\big] & \text{if } x \in B(x_0, D) \setminus B(x_0, D/2), \\ d & \text{if } x \in B(x_0, D/2), \end{cases}$$

$r_1 := \frac{1}{p}(\frac{c_1}{k})^p$ and $r_2 := \frac{1}{p}(\frac{c_2}{k})^p$.

We have

$$\begin{aligned}\Phi(u_d) &= \frac{1}{p}\|u_d\|^p = \frac{1}{p}\int_\Omega \big|\nabla u_d(x)\big|^p\,dx = \frac{1}{p}\int_{B(x_0,D)\setminus B(x_0,D/2)} \frac{(2d)^p}{D^p}\,dx \\ &= \frac{1}{p}\frac{(2d)^p}{D^p}\big(m\big(B(x_0,D)\big) - m\big(B(x_0,D/2)\big)\big) \\ &= \frac{1}{p}\frac{(2d)^p}{D^p}\frac{\pi^{N/2}}{\Gamma(1+N/2)}\big(D^N - (D/2)^N\big) = \frac{1}{p}\frac{K^p}{k^p}d^p.\end{aligned}$$

Now, taking into account that from the compact embedding of X in $C(\overline{\Omega})$ one has $\max_{x\in\overline{\Omega}}|u(x)| \leqslant c_1$ for all $u \in X$ such that $\Phi(u) < r_1$, we obtain

$$\frac{\chi(r_1)}{r_1} = \frac{\sup_{\Phi(u)<r_1}\Upsilon(u)}{r_1} \leqslant m(\Omega)pk^p\frac{\sup_{|\xi|\leqslant c_1}F(\xi)}{c_1^p} \tag{4.18}$$

and, in a similar way, we also obtain

$$\frac{\chi(r_2)}{r_2} = \frac{\sup_{\Phi(u)<r_2}\Upsilon(u)}{r_2} \leqslant m(\Omega)pk^p\frac{\sup_{|\xi|\leqslant c_2}F(\xi)}{c_2^p}. \tag{4.19}$$

Moreover, since $F(t) \geqslant 0$ for all $t \in [0, d]$, we obtain

$$\Upsilon(u_d) \geqslant \int\limits_{B(x_0, D/2)} F(d)\,dx = F(d)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\frac{D^N}{2^N}.$$

Hence, one has

$$\frac{\Upsilon(u_d)}{\Phi(u_d)} \geqslant \frac{pk^p}{K^p}\frac{\pi^{N/2}}{\Gamma(1+N/2)}\frac{D^N}{2^N}\frac{F(d)}{d^p} = p\frac{D^p}{2^p(2^N-1)}\frac{F(d)}{d^p}. \tag{4.20}$$

Therefore, from (4.18)–(4.20) and (h_1) the assumptions (a_1'') and (a_2') of Corollary 3.1 follow.

Now, we claim that the generalized critical points of $\Phi - \lambda\Upsilon$ are weak solutions for the problem (D_λ). To this end, let $u_0 \in X$, comply with $(\Phi - \lambda\Upsilon)^0(u_0, v - u_0) \geqslant 0$ for all $v \in X$. In particular, one has

$$\Phi'(u_0)(w) + \lambda(-\Upsilon)^0(u_0, w) \geqslant 0 \quad \text{for all } w \in X,$$

$$\Phi'(u_0)(w) \geqslant -\lambda(-\Upsilon)^0(u_0, w) \quad \text{for all } w \in X,$$

$$-\int\limits_\Omega \bigl|\nabla u_0(x)\bigr|^{p-2}\nabla u_0(x)\nabla w(x)\,dx \leqslant \lambda(-\Upsilon)^0(u_0, w) \tag{4.21}$$

for all $w \in X$. Clearly, setting $L(w) = -\int_\Omega |\nabla u_0(x)|^{p-2}\nabla u_0(x)\nabla w(x)\,dx$ for all $w \in X$, L is a continuous and linear functional on X; for which, (4.21) signifies $L \in \lambda\partial(-\Upsilon)(u_0)$. Taking into account that X is dense in $L^p(\Omega)$, from [16, Theorem 2.2] one has $L(w) \leqslant \lambda(-\Upsilon)^0(u_0, w)$ for all $w \in L^p(\Omega)$, so that L is a continuous and linear functional on $L^p(\Omega)$. Therefore, there is $\bar{u} \in L^{\frac{p}{p-1}}(\Omega)$ such that $L(w) = \int_\Omega \bar{u}(x)w(x)\,dx$ for all $w \in L^p(\Omega)$ and, in particular, $L(w) = \int_\Omega \bar{u}(x)w(x)\,dx$ for all $w \in X$. Hence, $-\int_\Omega |\nabla u_0(x)|^{p-2}\nabla u_0(x)\nabla w(x)\,dx = \int_\Omega \bar{u}(x)w(x)\,dx$ for all $w \in X$; so, $\bar{u}$ is the weak derivative of $|\nabla u_0|^{p-2}\nabla u_0$, which is denoted by $\Delta_p u_0$, that is $\Delta_p u_0 = \bar{u}$. For which $\Delta_p u_0 \in L^{\frac{p}{p-1}}(\Omega)$ and, from (4.21), $\Delta_p u_0 \in \lambda\partial(-\Upsilon)(u_0)$. Hence, from [16, Theorem 2.1] we obtain

$$\Delta_p u_0(x) \in \lambda\bigl[(-f)^-\bigl(u_0(x)\bigr), (-f)^+\bigl(u_0(x)\bigr)\bigr] \quad \text{a.e. } x \in \Omega. \tag{4.22}$$

Now, since $m(D_f) = 0$, from Lemma 1 of [18] we obtain $\Delta_p u_0(x) = 0$ for almost all $x \in u_0^{-1}(D_f)$. Moreover, from (h_2) we obtain $-f(u_0(x)) = 0$ for all $x \in u_0^{-1}(D_f)$. Hence,

$$-\Delta_p u_0(x) = 0 = f\bigl(u_0(x)\bigr)$$

for almost all $x \in u_0^{-1}(D_f)$.

Therefore, since at almost all $x \in \Omega \setminus u_0^{-1}(D_f)$ condition (4.22) reduces to

$$-\Delta_p u_0(x) = \lambda f\bigl(u_0(x)\bigr),$$

our claim is proved.

Finally, from the Weak Maximum Principle (see, for instance, [19]) the assumption (3′) of Corollary 3.1 (see Remark 3.9) follows. In fact, let u_1 and u_2 be two local minima for $\Phi - \lambda\Upsilon$. Then, as seen before, they are weak solutions for the problem (D_λ) and, since f is non-negative, they are non-negative functions. Hence, one has $\Upsilon(tu_1 + (1-t)u_2) \geqslant 0$ for all $t \in [0,1]$.

Hence, owing to Corollary 3.1, for each $\lambda \in \Lambda$, the functional $\Phi - \lambda\Upsilon$ admits at least three critical points u_i, $i = 1,2,3$, whose norms are less than $(pr_2)^{\frac{1}{p}}$, that is the problem (D_λ) admits three (non-negative) weak solutions u_i, $i = 1,2,3$, such that $\max_{x\in\Omega} |u_i(x)| \leqslant k\|u_i\| < c_2$ and the assertion follows. □

Remark 4.6. Clearly, Theorem 1.1 in the Introduction is a consequence of Theorem 4.4. We also observe that considerations such as those in Remarks 4.3–4.5 can be made. In particular, we just point out that, when the function f is non-negative, Theorem 4.4 improves [11, Theorem 3.1].

Remark 4.7. We explicitly observe that Theorem 3.3, also taking Remarks 3.10 and 3.11 into account, can be directly applied to nonlinear differential problems with continuous data (see also [3, Theorem 5.1] and Remark 3.3) as, for instance, those studied in [4,9,12,13,23,24,37,41,42] owing to critical points theorems established in [2,5,6,8], and, hence, in these problems the existence of three solutions without any asymptotic condition can be investigated.

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